8/26/19

DRW Questions

Exit Times

11)

$$\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & & & & \\ 1 & & & & \\ 2 & & & & \\ 3 & & & & \end{array}
\quad
\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array}
\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{5}{6} & \frac{1}{6} & 0 \\ 0 & \frac{5}{6} & 0 & \frac{1}{6} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$g(\cdot)$: expected time I spend in state i

$g(3) = 0$

$$\begin{cases} g(0) = 1 + 1 \cdot g(1) \\ g(1) = 1 + \frac{5}{6} g(1) + \frac{1}{6} g(2) \\ g(2) = 1 + \frac{5}{6} g(1) + \underbrace{\cancel{\tfrac{1}{6} g(3)}}_{=0} \end{cases}$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & \frac{1}{6} & -\frac{1}{6} \\ 0 & -\frac{5}{6} & 1 \end{bmatrix} \begin{bmatrix} g(0) \\ g(1) \\ g(2) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad g(0) = 43$$

Binomial Tree Model

Consider the return on a stock price w/ n steps

$$K_n = \begin{cases} u & \text{w/ prob } p \\ d & \text{w/ prob } 1-p \end{cases}$$

where $-1 < d < u$ and $0 < p < 1$

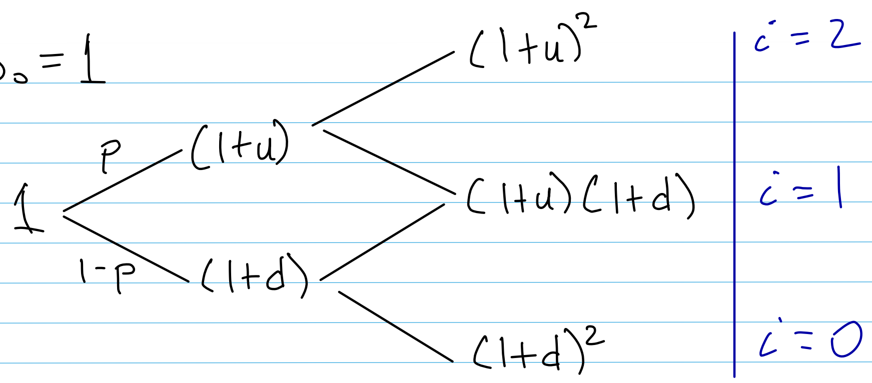
- S_n can move $(1+u)$ and $(1+d)$ at each time step
- S_0 as initial stock price, S_1 stock price at $n=1$
- $\frac{S_1}{S_0} = 1 + k_1$

$$S_1 = \begin{cases} S_0(1+u) & \text{w/ prob. } p \\ S_0(1+d) & \text{w/ prob. } 1-p \end{cases}$$

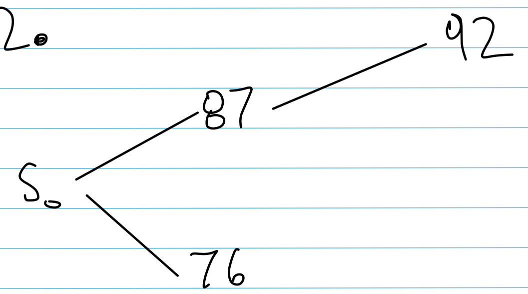
- As a result:

$$S_n = S_0 (1+u)^i (1+d)^{n-i} \quad \text{w/ prob.} \quad \binom{n}{i} p^i (1-p)^{n-i}$$

Let $S_0 = 1$



Ex) Find d and u if S_1 can take \$87 or \$76, and the top possible value S_2 is \$92.



Risk-neutral Probability

The expected one-step return can be written as:

$$\mathbb{E}[K_1] = p(u) + (1-p)d$$

- We'll introduce a special symbol p^* to denote the risk-neutral probability

$$\mathbb{E}^*[K_1] = \underline{p^* u + (1-p^*)d = r}$$

$$p^* = \frac{r-d}{u-d}$$

Value of a Portfolio

$$V_n = \sum_{i=0}^{d} \theta_n^i \cdot S_n^i \quad \text{where} \quad \theta_n^i = \text{\# of shares invest in asset } i \text{ at step } n$$

- We assume S_t^0 is our riskless asset

The initial wealth is denoted by

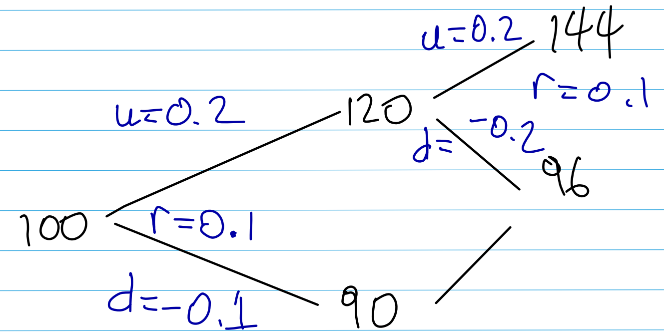
$$V_0 = \sum_{i=0}^{d} \theta_0^i S_0^i$$

No arbitrage principle

- There is no strategy s.t. $V_0 = 0$ and $V_n > 0$ w/ positive probability for some $n > 0$.
- The binomial tree model admits no arbitrage iff $d < r < u$

Ex) Consider a market w/ risk-free asset s.t. $S_0^0 = 100$, $S_1^0 = 110$, $S_2^0 = 121$ and risky asset w/ price process

Scenario	S_0^1	S_1^1	S_2^1
ω_1	100	120	144
ω_2	100	120	96
ω_3	100	90	96



$S_1^1 = 90$, $S_1^0 = 110$

Is there an arbitrage oppurtunity if

a) If there are no restrictions on short-selling?

b) No short-selling is allowed.

$(\theta_1^0, \theta_1^1) = (x, y)$

$$\boxed{\begin{array}{l} n=1 \\ n=2 \end{array} \begin{cases} 110x + 90y = 0 \\ 121x + 96y > 0 \end{cases}}$$

$$x = \frac{-90y}{110}$$

$$121\left(-\frac{90}{110}y\right) + 96y > 0$$

$$-3y > 0$$

Let $y = -\frac{1}{3}$

$$x = \frac{-90}{110}\left(-\frac{1}{3}\right)$$

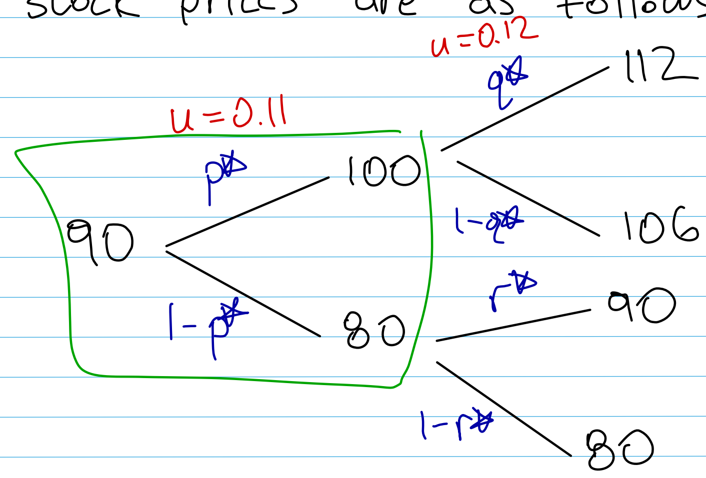
$n=1$: $110\left(\frac{3}{11}\right) + 90\left(-\frac{1}{3}\right) = 0$

$n=2$: $121\left(\frac{3}{11}\right) + 96\left(-\frac{1}{3}\right) = 1$

Option Pricing

Consider the model, $S_0^0 = 100$, $S_1^0 = 110$, $S_2^0 = 121$ and suppose that the stock prices are as follows:

$r = 0.1$

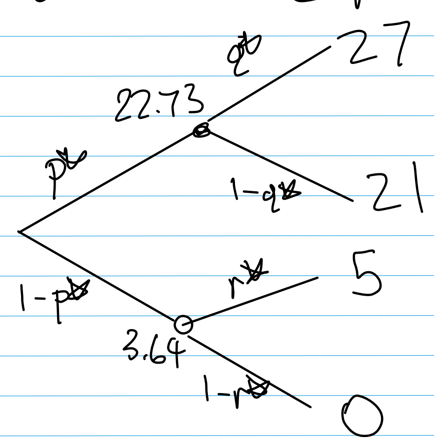


$$\begin{cases} 100p^* + 80(1-p^*) = 90(1.1) \\ 112q^* + 106(1-q^*) = 100(1.1) \\ 90r^* + 80(1-r^*) = 80(1.1) \end{cases} \Rightarrow (p^*, q^*, r^*) = \left(\frac{19}{20}, \frac{2}{3}, \frac{4}{5}\right)$$

Suppose we have a European Call option that pays:

$$C_2^E = \max\{S_2^1 - \underset{\text{strike price}}{85}, 0\}$$

In other words, the payoff of call option



Then we can find the price of this option:

$r=0.1$

$$C_1^E(\omega_1) = \frac{1}{(1+r)}[27q^* + 21(1-q^*)] = 22.73$$

$$C_1^E(\omega_2) = \frac{1}{1+r}[5r^* + 0(1-r^*)] = 3.64$$

$$C_0^E = \frac{1}{1+r}[22.73p^* + 3.64(1-p^*)] = 19.79$$

Ex) Calculate value of a Put Option: $P_2^E = \max\{100 - S_2^1, 0\}$ w/ the same parameters as before

Put-Call Parity

For European Call and Put options w/ the same strike price X and exercise time T.

$$C^E - P^E = S_1^0 - Xe^{-rT}$$

Otherwise an arbitrage opportunity exists.

Bounds on Option Prices

$$C^E \leq C^A \quad , \quad P^E \leq P^A$$

Ex) The American option can be exercised at any time step n w/ payoff $f(S_n)$

At expiry: $C_n^A = f(S_n)$, Suppose $n=2$

At time 1, the option holder can exercise immediately w/ $f(S_1)$ or wait until time step 2

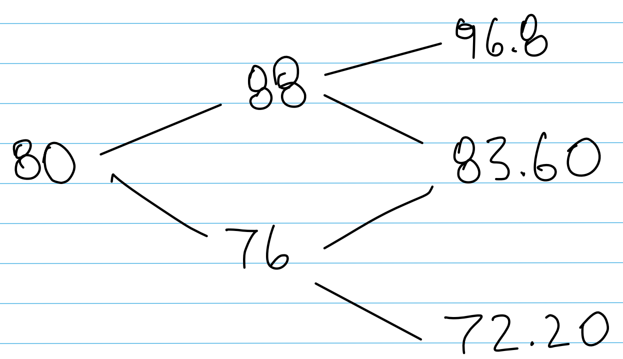
- This means at time 1,

$$C_1^A = \max\left\{ f(S_1), \frac{1}{1+r}\left[p^* f(S_1(1+u)) + (1-p^*) f(S_1(1+d))\right]\right\}$$

And Similarly for C_0^A

- Consider an American Put option w/ strike Price $X = 80$ that expires at $n=2$

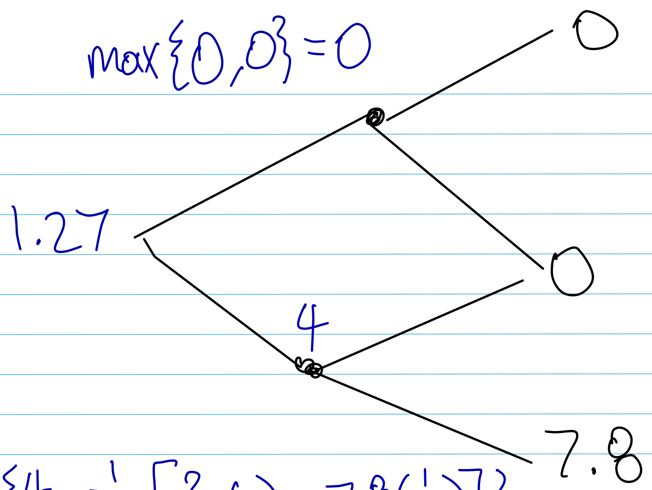
$S_0^1 = 80$, $u = 0.1$, $d = -0.05$, $r = 0.05$



$$p^* = \frac{r-d}{u-d} = \frac{2}{3}$$

$\max\{0,0\}=0$

$C_0^A = \max\{0, \frac{1}{1.05}[0(\frac{2}{3})+(\frac{1}{3})(4)\} = 1.27$



$\max\{4, \frac{1}{1.05}[\frac{2}{3}(0)+7.8(\frac{1}{3})]\}$
$=2.48$

Ex) Compute the value of an American Call w/ strike $X=120$ at time 2 initial price $S_0^1=120$, $u=0.2$, $d=-0.1$, and $r=0.1$