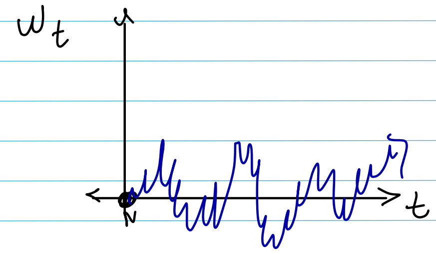
8/28/19

Continuous time Finance

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. The process "W" will be said Brownian Motion if:



i) $W_0 = 0$

ii) W_t has continuous paths

iii) The process W has independent increments, i.e. $r < s \le t < u$ then $W_u - W_t$ and $W_s - W_r$ are independent stochastic variables,

iv) For every $0 \le s < t$, $W_t - W_s \sim N(0, t-s)$

or $W_t \sim N(0, t)$

Ex) The random variable W_t and $\sqrt{t} \cdot Z$ where $Z \sim N(0,1)$. Have the same distribution but $\sqrt{t} \cdot Z$ is not BM.

- Assume $0 \le s < t$, $X_t = \sqrt{t} \cdot Z$, then $X_t - X_s = (\sqrt{t} - \sqrt{s}) \cdot Z$

$\mathbb{E}[X_t - X_s] = 0$

$Var(X_t - X_s) = (\sqrt{t} - \sqrt{s})^2 \cdot Var(Z) = t + s - 2\sqrt{ts}.$

- Brownian Motions not differentiable at any time t for almost all paths.

Martingales

A stochastic process $X = \{X_t\}_{t \in T}$ is a martingale for the filtration $(\mathcal{F}_t)_{t \in T}$ if

a) X is adapted to $(\mathcal{F}_t)_{t \in T}$

b) X_t is integrable for each t

c) for $s \le t$, $\mathbb{E}[X_t \mid \mathcal{F}_s] = X_s$

- In particular for $0 < s \le t < u$, the increment $W_u - W_t$ is independent of $\mathcal{F}_t^W$

Ex) Each of the following are martingales w.r.t. $\mathcal{F}_t^W$,

a) W_t

b) $W_t^2 - t$

c) $\exp\{\sigma W_t - \frac{1}{2}\sigma^2 t\}$

Asset Dynamics

The market contains two securities

- The risk-free asset,

$$dB_t = r \cdot B_t \, dt \quad \text{i.e.} \quad dy = r \cdot y \, dt$$

w/ $B_0 = 1$, $r > 0$, solving this ODE

$$B_t = e^{rt}$$

- The risky asset is represented by (S_t),

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t \quad \text{(SDE)}$$

where S_0 is given, $\mu \in \mathbb{R}$ is the drift and $\sigma > 0$ is the volatility of stock price S.

- The solution to this SDE is given by,

$$S_t = S_0 \cdot \exp\{\mu t - \frac{\sigma^2}{2} t + \sigma W_t\}$$

Itô's Formula

- Assume X has SDE given by,

$$dX_t = \mu \, dt + \sigma_t \, dW_t$$

- Define $Z = f(t, X_t)$, then the SDE of Z is given by,

$$dZ = df(t, X_t) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} dX + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} (dX)^2$$

where,

$$\begin{cases} (dt)^2 = 0 \\ dt \cdot dW = 0 \\ (dW)^2 = dt \end{cases}$$

$$df(t, X_t) = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} (\mu \, dt + \sigma_t \, dW_t) + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \sigma_t^2 \, dt$$

$$(dX)^2 = (\mu \, dt + \sigma_t \, dW_t)(\mu \, dt + \sigma_t \, dW_t)$$

$$= \cancel{\mu^2 (dt)^2} + \cancel{\mu \, dt \cdot \sigma_t \, dW_t} + \cancel{\sigma_t \, dW_t \, \mu \, dt} + \boxed{\sigma_t^2 \underbrace{(dW_t)^2}_{dt}}$$

$$df(t, X_t) = \{\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} \mu + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \sigma_t^2\} dt + \sigma_t \frac{\partial f}{\partial x} \, dW_t$$

Ex) Show by Itô's Formula that

$$S_t = S_0 \cdot \exp\{\mu t - \frac{\sigma^2}{2}t + \sigma W_t\}$$

solves

$$dS_t = \mu S_t\, dt + \sigma S_t\, dW_t$$

- Define $S_t := f(t, W_t)$, then

$$\frac{\partial f}{\partial t} = \underbrace{S_0 \cdot \exp\{\mu t - \frac{\sigma^2}{2}t + \sigma W_t\}}_{S_t} \cdot (\mu - \frac{\sigma^2}{2}) \cdot dt$$

$$\frac{\partial f}{\partial W} = \underbrace{S_0 \cdot \exp\{\mu t - \frac{\sigma^2}{2}t + \sigma W_t\}}_{S_t} \cdot \sigma$$

$$\frac{\partial^2 f}{\partial W^2} = \underbrace{S_0 \cdot \exp\{\mu t - \frac{\sigma^2}{2}t + \sigma W_t\}}_{S_t} \cdot \sigma^2$$

$$dS_t = S_t(\mu - \frac{\sigma^2}{2}) \cdot dt + S_t \cdot \sigma \cdot dW_t + \frac{1}{2}\sigma^2 \cdot S_t \cdot \underbrace{(dW_t)^2}_{dt}$$

$$= \{S_t \mu - \cancel{S_t \frac{\sigma^2}{2}} + \cancel{\frac{1}{2}\sigma^2 S_t}\} dt + S_t \sigma dW_t$$

$$dS_t = \mu S_t\, dt + \sigma S_t\, dW_t$$

Ex) Compute the dynamics of $f(t, W_t) = W_t^2$

$$\frac{\partial f}{\partial t} = 0$$

$$\int_0^t W_s \cdot dW_s$$

$$\frac{\partial f}{\partial W} = 2W_t$$

$$\frac{\partial^2 f}{\partial W^2} = 2$$

$$df(t, W_t) = 0 + 2W_t \cdot dW_t + \frac{1}{2}(2) \cdot \underbrace{(dW_t)^2}_{dt}$$

$$\int d(W_t^2) = \int dt + 2W_t\, dW_t$$

$$W_t^2 = t + 2 \cdot \int_0^t W_s\, dW_s$$

$$\int_0^t W_s\, dW_s = \frac{1}{2}(W_t^2 - t)$$

Ex) Compute the dynamics of $f(t, W_t) = t \cdot W_t$

$$\frac{\partial f}{\partial t} = W_t \, dt$$

$$\frac{\partial f}{\partial W} = t$$

$$\frac{\partial^2 f}{\partial W^2} = 0$$

$$\int d(t \cdot W_t) = \int W_t \, dt + t \cdot dW_t$$

$$t \cdot W_t = \int_0^t W_s \, ds + \int_0^t s \, dW_s$$

$$\int_0^t W_s \, ds = t W_t - \int_0^t s \, dW_s$$

Explicit Formula for Call Option and Put Option

$$C_t = \max\{S_t - K, 0\}$$

Value is given by,

$$F(t,s) = e^{-\delta(T-t)} \cdot S \cdot N[d_1(t,s)] - e^{-r(T-t)} \cdot K \cdot N[d_2(t,s)]$$

$$d_{\underset{2}{1}} = \frac{\log\left(\frac{S}{K}\right) + \left(r - \delta \mp \frac{\sigma^2}{2}\right)(T-t)}{\sigma \cdot \sqrt{T-t}}$$

δ: continuous dividend yield rate,

K: strike price

$N[\cdot]$: CDF of a $N(0,1)$

T : time of expiry

t : current time

σ: Volatility of the asset

r: risk-free rate

S: underlying price S_t

For $\delta = 0$,

- Price of a Call Option,

$$P = F(t,s) = S \cdot N[d_1(t,s)] - e^{-r(T-t)} \cdot N[d_2(t,s)]$$

$$d_{1,2} = \frac{\log\left(\frac{S}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

- Price of a Put Option,

$$P = F(t,s) = e^{-r(T-t)} \cdot K \cdot N[-d_2(t,s)] - S \cdot N[-d_1(t,s)]$$

The Greeks

"delta" $\Delta P = \frac{\partial P}{\partial S}$ The ratio of the chang in price of the option w.r.t. the underlying. Also called the hedge ratio

"gamma" $\Gamma P = \frac{\partial^2 P}{\partial S^2}$

"rho" $\rho P = \frac{\partial P}{\partial r}$

"theta" $\Theta P = \frac{\partial P}{\partial t}$

"vega" $\mathcal{V}_P = \frac{\partial P}{\partial \sigma}$ ← change in portfolio w.r.t. volatility

Ex) Delta of a Call option

$$\frac{\partial P}{\partial S} = \frac{\partial F(t,S)}{\partial S} = 1 \cdot N[d_1(t,S)] + S \cdot \varphi[d_1(t,S)] \cdot \frac{\partial d_1}{\partial S}$$

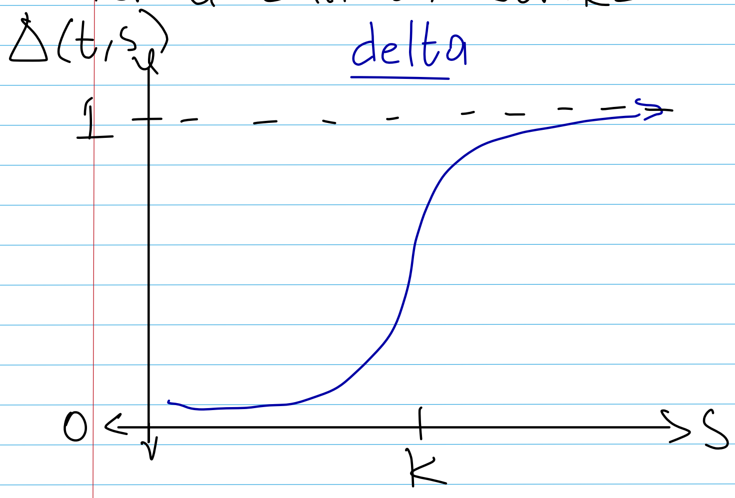
$$\frac{\partial d_1}{\partial S} = \frac{1}{\left(\frac{S}{K}\right)} \cdot \left(\frac{1}{K}\right) \cdot \frac{1}{\sigma\sqrt{T-t}} = \frac{1}{S \cdot \sigma\sqrt{T-t}}$$

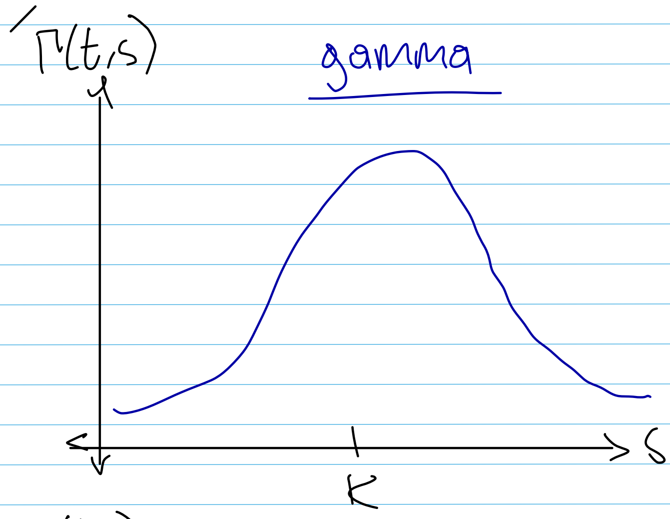
$$- e^{-r(T-t)} \cdot \varphi(d_2(t,S)) \cdot \frac{\partial d_2}{\partial S}$$

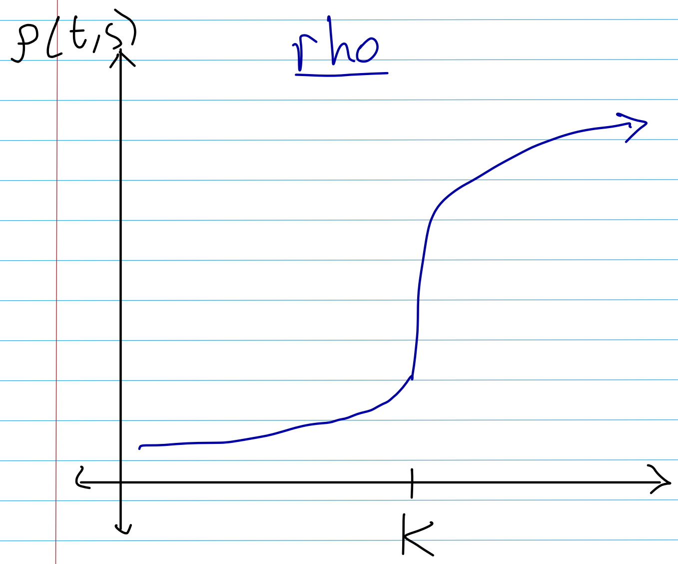
$$\frac{\partial P}{\partial S} = N[d_1(t,S)] + \underbrace{S\varphi[d_1(t,S)] \cdot \frac{1}{S \cdot \sigma\sqrt{T-t}} - e^{-r(T-t)} \cdot \varphi[d_2(t,S)] \cdot \frac{1}{S \cdot \sigma\sqrt{T-t}}}_{=0}$$

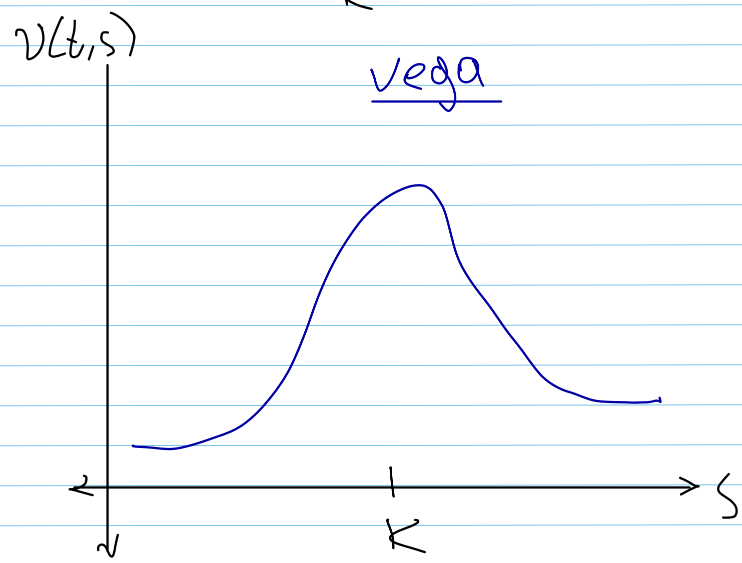
Ans: $\frac{\partial P}{\partial S} = N[d_1(t,S)]$

For a Call w/ strike K,



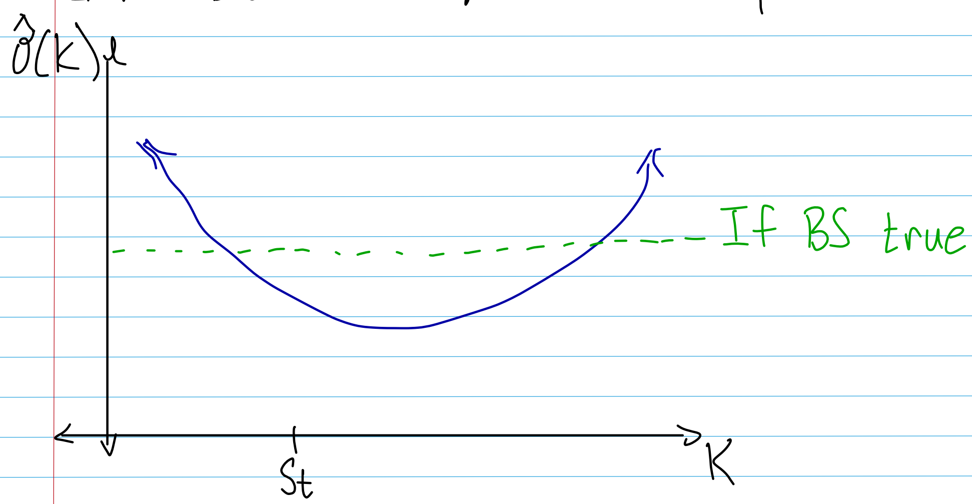






Is historical volatility = implied volatility?

- If BS was a perfect description of reality, then true.



Ex) A 6-month call option w/ an exercise price of \$50 is currently trading at \$52, costs \$4.5. Determine whether you should buy this option if the risk-free rate is 5% and the annual std. deviation is 12%.

- Approximation for Call option w/ strike $K = S \cdot e^{r(T-t)}$

$$F(t,s) = s \cdot N\left[\frac{\sigma}{2} \cdot \sqrt{T-t}\right] - S \cdot N\left[-\frac{\sigma}{2} \cdot \sqrt{T-t}\right]$$

$$d_1 = \frac{\log\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} = \frac{\cancel{-r(T-t)} + \left(\cancel{r} + \frac{\sigma^2}{2}\right)\cancel{(T-t)}}{\cancel{\sigma}\sqrt{T-t}}$$

$$\cancel{\log}\left(\cancel{e}^{-r(T-t)}\right) \qquad = \frac{\sigma}{2}\sqrt{(T-t)}$$

$$d_2 = -\frac{\sigma}{2}\sqrt{T-t}$$

First order Taylor expansion of the Normal CDF : $N(x)$

$$N(x) \approx N(0) + \varphi'(0) \cdot x$$

$$F(t,s) \approx \cancel{s \cdot [N(0)} + \varphi'(0) \cdot \frac{\sigma}{2}\sqrt{T-t}] - \cancel{S[N(0)} + \varphi'(0) \cdot \left(-\frac{\sigma}{2}\sqrt{T-t}\right)]$$

$$\approx \varphi'(0)\, \sigma\sqrt{T-t}\,(s) \quad , \quad \varphi'(0) \approx 0.4$$

$$\approx 0.4 \cdot \sigma \cdot s\sqrt{T-t}$$